Synchronization in coupled map lattices as an interface depinning

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We study a solid-on-solid (SOS) model whose dynamics is inspired by recent studies of the synchronization transition in coupled map lattices (CML). The synchronization of CML is thus related with a depinning of interface from a binding wall. Critical behavior of our SOS model depends on a specific form of binding (i.e., transition rates of the dynamics). For an exponentially decaying binding the depinning belongs to the directed percolation universality class. Other types of depinning, including the one with a line of critical points, are observed for a power-law binding.

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Recently synchronization of chaotic systems has attracted a lot of interest [1]. To a large extent, this interest is motivated by numerous experimental realizations of this phenomena including lasers, electronic circuits, or chemical reactions [2]. Synchronization acquires additional features in spatially extended systems, where it can be regarded as a certain nonequilibrium phase transition. There are increasing efforts to understand the properties of this transition. Relatively well understood is the synchronization transition (ST) in certain cellular automata. Since a synchronized state can be regarded as an absorbing state for cellular automata, which are discrete systems, the phase transition as expected, belongs to the directed percolation (DP) universality class [3]. However, continuous systems, as, e.g., coupled map lattices (CML's) [4], need infinite time to reach a synchronized state and the relation with DP does not seem to hold. Indeed, Pikovsky and Kurths [5] argued that for continuous systems this transition should belong to the so-called bounded Kardar-Parisi-Zheng (BKPZ) universality class [6]. Recently, precise numerical calculations confirmed their predictions but only for CML with some continuous maps [7]. Surprisingly, the ST for discontinuous [7,8] or continuous but sufficiently steep [9] maps were found to belong to the DP universality class. It would be desirable to understand the critical behavior of the ST in CML and the present paper might be a step in this direction.

Let us briefly describe the setup which is used to study synchronization in CML [7]. In the simplest one-dimensional case, one takes a single-chain CML of size L that is composed of L diffusively coupled local maps $f(u_i)$ [4]. The maps are chaotic and act on continuous site variables u_i ($i=1,\ldots,L$), which are typically bounded ($0<u_i<1$). Then, one couples such a spatio-temporally chaotic system with its identical copy, which initially has a different set of site variables. It turns out that the evolution of such a coupled system depends on the coupling strength. For weak coupling the two CML's are desynchronized and essentially independent. However, for sufficiently strong coupling the system gets synchronized and approaches a state where corresponding pairs of site variables in both copies take the same value. To quantify synchronization one can introduce the synchronization error $w(i,t)=|u_1(i,t)-u_2(i,t)|$ where a lower index denotes a copy of CML and its spatial average $w(t)=1/L\Sigma_{i=1}^{L}w(i,t)$. To relate this problem with BKPZ one argues that in continuous limit and close to the synchronized state, the evolution equation for $w(i,t)$ is given as a Langevin equation with multiplicative Gaussian noise which then, using the Hopf-Cole transformation $h=-\ln(w)$, is transformed into BKPZ. Let us notice that CML's are deterministic systems and the noise has only effective meaning, mimicking their chaotic behavior. The problem of a relation of such deterministic systems with stochastic counterparts is very interesting and recently is drawing some attention [10]. In the above representation the desynchronized phase in CML ($w(t)>0$) corresponds to the interface pinned relatively close to the wall ($\langle h_i\rangle<\infty$). In the synchronized phase $[w(t)\to 0]$ the interface depins and drifts away ($\langle h_i\rangle\to\infty$). Perfectly synchronized state $[\langle w(t)\rangle=0]$ is reached only after infinitely long time. The above analysis requires the differentiability of the local map and thus is not applicable to discontinuous maps. However, numerical results show that the relation with BKPZ breaks down also for continuous but sufficiently steep maps [9]. From a theoretical point of view it would be desirable to understand why such properties of the local map affect the nature of the ST and move into the DP universality class. Let us notice that if the coupled CML system enters the synchronized state, it will remain in this state forever. Such a state can thus be considered as an absorbing state of the dynamics, although it cannot be reached in any finite time. Well-developed techniques are available to study phase transitions in models with absorbing states [11].

It is clear that the problem of synchronization in extended systems are related with a number of very interesting problems in nonequilibrium statistical mechanics such as the KPZ model, nonequilibrium wetting [12], or directed percolation. Recently, some arguments were given that a notoriously difficult particle system, the so-called pair contact process with diffusion (PCPD) model, might be also related with these models [13]. Deeper understanding of these problems and their mutual relations would be certainly desirable.

In the present paper we introduce an interfacial and discrete (SOS) model which is inspired by the dynamics of CML with discontinuous maps. The synchronization of CML is thus related with a depinning of interface from a binding wall. Numerical calculations show that the universality class of the depinning transition in our model depends on the choice of binding, which enters the dynamics through certain transition rates. For an exponentially decaying binding, the depinning belongs to the DP universality class. In this case the overall behavior of the model is very similar to Sneppen's model of interface propagation in a random environment that is driven by a certain extremal dynamics [14,15]. For a power-law decaying binding, the depinning transition is characterized by a different set of critical exponents. In the case of a rapid decay these exponents are very close to those obtained for the bosonic version of PCPD model [16]. In the case of a slow decay, in the entire unbounded phase the interface remains critical, that in itself is an interesting property of a nonequilibrium system.

In our model discrete site variables $h_i=1,2,\dots$ are defined on a one-dimensional lattice of size L $(i=1,\dots,L)$ with periodic boundary conditions $(h_{L+1}=h_1)$. In an elementary update, we select randomly a site i and change the variable h_i, and possibly neighboring ones, according to the following rule: (i) with probability $p(h_i)$ one sets $h_i=h_{i+1}=h_{i-1}=1$; (ii) with probability $1-p(h_i)$ the site variable h_i increases by unity $(h_i\rightarrow h_i+1)$. During a unit of time L elementary updates are performed. To complete the definition we have to specify the function $p(h)$. To allow a drift toward $h=\infty$, the function $p(h)$ must decay to zero for $h\rightarrow\infty$. Numerical results that we present below are obtained for two cases: $p(h)=\mathrm{e}^{-\gamma h}$ (model I) and $p(h)=a(h+1)^{-\gamma}$ (model II), where $\gamma>0$ and $a>0$ are control parameters of the model.

To make a link with synchronization in CML's, the following remarks are in order. Numerical calculations for CML's with discontinuous maps show that the Lyapunov exponent that governs the evolution of the synchronization error $w(i,t)$ is negative in the vicinity of the transition [7]. Approximately, the evolution of $w(i,t)$ is thus made of consecutive contractions $[w(i,t)\rightarrow cw(i,t)$ and $c<1]$ that, due to discontinuity of the map, are from time to time interrupted by discontinuous changes that might substantially increase the value of $w(i,t)$. To notice a link between the dynamics of synchronization error in CML and our model we introduce new variables $w_i=\mathrm{e}^{-h_i}$ (let us notice a similarity to the inverse Hopf-Cole transformation). Indeed, the increase of h_i by unity according to the rule (ii) decreases w_i by a factor e that corresponds to the contraction of $w(i,t)$. The first rule mimics the discontinuous jumps of $w(i,t)$. Since local maps in CML's are coupled, a jump at a site i also affects its neighbors.

Monte Carlo simulations of our model are similar to those of other models with absorbing states [11]. For some details related to the fact that the model needs an infinite time to reach an absorbing state see, e.g., Ref. [9]. We observed that for sufficiently large γ the interface depins from the $h=1$ wall and drifts away. For smaller γ the model remains in the active phase with the interface relatively close to the wall. To examine the nature of the phase transition, we introduced $w(t)=\langle 1/L\Sigma_{i=1}^{L}w(i,t)\rangle$ that in the steady state is

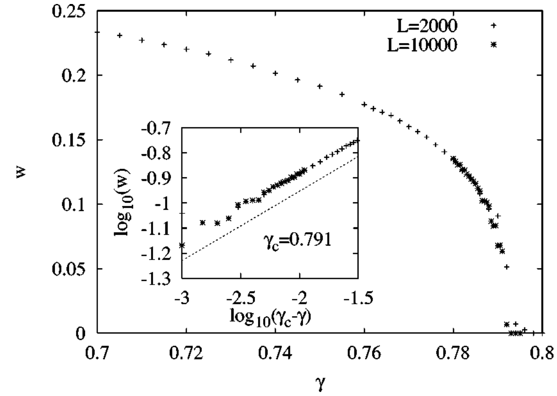

FIG. 1. The steady-state activity w as a function of γ for model I. The inset shows the logarithmic scaling of w at criticality. The slope of the dotted line corresponds to the DP value β=0.2765. The simulation time was $t_{\mathrm{sim}}=10^5$ plus 2×10^4 discarded for relaxation.

denoted as w (in the following we refer to this quantity as activity). Of course in the absorbing phase $w=0$ and in the active phase $w>0$. Upon approaching, the critical point w typically exhibits a power-law decay $w\sim(\gamma-\gamma_c)^{-\beta}$ with a characteristic exponent β and for the critical point located at $\gamma=\gamma_c$. Moreover, we studied the time dependence of $w(t)$. One expects that at criticality this quantity has a power-law decay $w(t)\sim t^{-\Theta}$, where Θ is another characteristic exponent. We also used the so-called spreading technique [17]. First, we set $h_i=\infty$ for all but one site (i_0) that was set to unity. Then we monitored the subsequent evolution of the model (actually, of interest are only sites with positive $w(i,t)$, i.e., with finite h_i) measuring the average activity of the system $w(t)$, the survival probability $P(t)$, and the averaged spread square $R^2(t)=1/w(t)\Sigma_i w(i,t)(i-i_0)^2$. One expects that at criticality: $w(t)\sim t^{\eta}$, $P(t)\sim t^{-\delta}$, and $R^2(t)\sim t^z$, which at the same time defines the critical exponents η, δ, and z.

First, let us describe the results obtained for model I

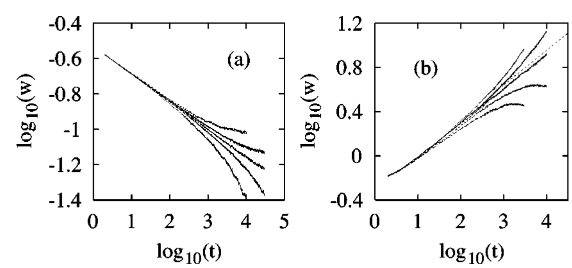

FIG. 2. (a) The time-dependent activity $w(t)$ as a function of time t for (from top) γ=0.788, 0.790, 0.791 (critical point), 0.792, and 0.794 (Model I). The results are averaged over 100 independent runs and simulations were done for $L=5\times10^4$. (b) The time-dependent activity $w(t)$ as a function of time t calculated using the spreading method for (from top) γ=0.787, 0.789, 0.791 (critical point), 0.793, and 0.795 (Model I). The results are averaged over 10^5 independent runs. The slope of the dotted line corresponds to the DP value η=0.3137

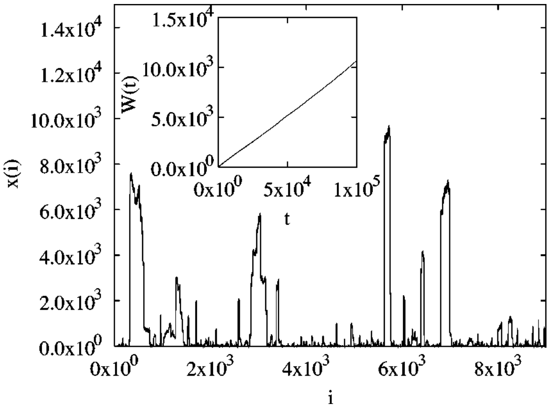

FIG. 3. The interface profile for model I at criticality ($\gamma=\gamma_c=0.791$) after $t=10^4$ Monte Carlo steps. The inset shows that the interface width $W(t)$ grows linearly in time.

$[p(h)=e^{-\gamma h}]$. From the steady-state measurements of w (Fig. 1), we estimate $\gamma_c=0.791(1)$ and $\beta=0.28(1)$. Such a location of the critical point is confirmed from time-dependent simulations [Fig. 2(a)] and the spreading method [Fig. 2(b)]. From these data we estimate $\Theta=0.160(2)$, $\eta=0.317(5)$, $\delta=0.16(1)$, and $z=1.26(1)$. The results for $P(t)$ and $R^2(t)$ are not presented. Obtained values of critical exponents clearly show that model I belongs to the DP universality class for which [11] $\beta=0.2765$, $\eta=0.3137$, $\Theta=\delta=0.1595$, and $z=1.265$.

To have a more complete insight into the critical behavior of our model we calculate the average interfacial width $W(t)=\langle(\Sigma_{i=1}^{L}[h(i,t)-\langle h(i,t)\rangle])^2\rangle^{1/2}$. At criticality this quantity typically behaves as $W(t)\sim t^{\beta'}$, where β' is the growth exponent. For model I simulations show almost linear increase of $W(t)$ with time (inset in Fig. 3) and we estimate $\beta'=1.0(1)$. Such a value of the growth exponent indicates very strong fluctuations, which is confirmed through a visual inspection of the interface profile (Fig. 3). In principle, from direct calculations for CML's we can obtain an interface profile as a logarithm of the synchronization error. Our preliminary calculations for CML model examined by Ahlers and Pikovsky [7] show that large fluctuations, similar to those in model I are clearly seen. However, to quantify these fluctuations and calculate, e.g., the width $W(t)$, simulations on a longer time scale are required. Since for CML, dynamics is defined in terms of w's rather than h's, simulations for longer time, which must probe states of extremely small synchronization error, severely suffer from the finite accuracy of numerical computations. On the other hand our SOS model easily allows us to examine such a long-time regime.

It is already known that some SOS-like models belong to the DP universality class. In particular, Alon *et al.* [18] introduced a model of interface roughening with edge evaporation. In this model the growth of the interface width is logarithmic in time and is thus much different from that of our model. As far as the critical behavior is concerned, our model seems to be more closely related with the Sneppen's

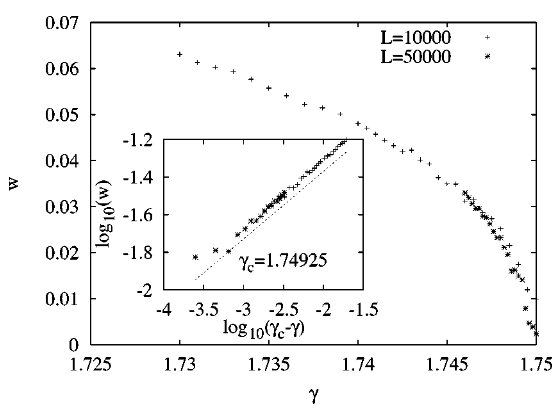

FIG. 4. The steady-state activity w as a function of γ for model II for $a=1$. The inset shows the logarithmic scaling of w at criticality. The slope of the dotted line corresponds to $\beta=0.36$. The simulation time was $t_{\rm sim}=10^5$ plus 2×10^4 discarded for relaxation.

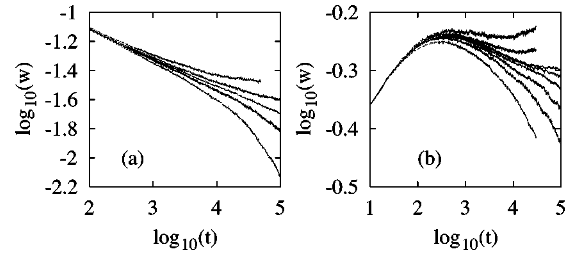

FIG. 5. (a) The time-dependent activity $w(t)$ as a function of time t for (from top) $\gamma=1.746$, 1.748, 1.74925 (critical point), 1.750, and 1.752 (Model II, $a=1$). The results are averaged over 100 independent runs and simulations were done for $L=5\times10^4$. (b) The time-dependent activity $w(t)$ as a function of time t calculated using the spreading method for (from top) $\gamma=1.7488$, 1.749, 1.7492, 1.74925 (critical point), 1.7493, 1.7494, 1.7496, and 1.75 (Model II, $a=1$). The results are averaged over 10^5 independent runs.

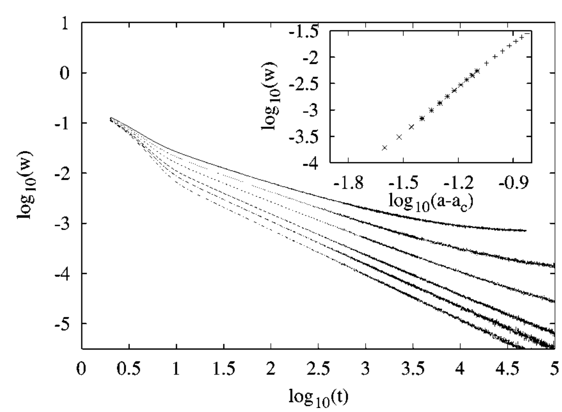

FIG. 6. The time-dependent activity $w(t)$ as a function of time t for (from top) $a=0.11$, 0.09, 0.07 (critical point), 0.05, and 0.04 (Model II, $\gamma=1$). The results are averaged over 200 independent runs and simulations were done for $L=5\times10^4$. Inset shows the logarithmic scaling of the steady-state activity w as a function of $a-a_c$ with $a_c=0.07$ calculated for $L=2\times10^4(+)$ and $L=5\times10^4(\times)$ (Model II, $\gamma=1$).

model of interface spreading in environment with quenched disorder [14]. His model is driven by extremal dynamics and has the growth exponent β' close to unity. Later it was established that this model is actually equivalent to the DP model sitting exactly at the critical point [15]. Important ingredients of Sneppen's model are quenched disorder and extremal dynamics. The fact that our model, which misses these features, exhibits essentially the same behavior is in our opinion quite interesting and worth further examination.

Using the same procedure we studied the model II ($P(h)=a(h+1)^{-\gamma}$). First we kept $a=1$ fixed and varied only the parameter γ. Some of our numerical results are shown in Figs. 4 and 5. Using these data we estimate $\gamma_c=1.74925(5)$, $\beta=0.36(1)$, $\Theta=0.185(3)$, $\eta=-0.03(1)$, $\delta=0.445(5)$, and $z=1.19(1)$. All exponents considerably differ from DP exponents. To support these estimations, let us notice that the hyperscaling relation $z/2=\Theta+\delta+\eta$ is satisfied by the above values. It is rather surprising for us to observe that our values of exponents β, Θ, and z are in a very good agreement with recent estimation for the so-called bosonic version of a pair contact process with diffusion [16]. It would be interesting to examine whether this is only a numerical coincidence or if there is a deeper relation between these two problems.

We also studied model II for fixed γ and varying the amplitude a. Monte Carlo simulations show that for $\gamma=2$ the critical behavior of the model seems to be the same as for $\gamma=\gamma_c=1.74925$ and $a=1$. It suggests that for a certain range of γ the depinning transition belongs to the same universality class. However, a different behavior was observed for $\gamma=1$. Indeed, in this case our simulations suggest (Fig. 6) that the depinning transition is characterized by the exponents $\beta\sim3.0(3)$ and $\Theta=0.62(5)$. What is also interesting, the entire unbounded phase $a<a_c\sim0.07$ seems to be critical with a power-law decaying order parameter $w\sim t^{-x}$, where $x\sim0.80(5)$, and x might slightly vary with a. Such a behavior is also worth further studies.

One of the future problems would be to check whether there are some other types of critical behavior for this kind of SOS model. Clearly, an important ingredient that determines the critical behavior is the form of the function $p(h)$. It would be interesting to explore some other functions (e.g., $e^{-\gamma h^2}$ or $[\ln(h)]^{-\gamma}$) for a possibly new behavior. Although we were not able to recover the BKPZ universality class within our approach, there is still a possibility that for a certain choice of $p(h)$ such a behavior might appear. It would be also interesting to check whether a new critical behavior found in model II has a counterpart in a synchronization transition in CML.

Note added in proof. Recently, another stochastic model with dynamics inspired by the synchronization phenomenon was examined by Ginelli *et al.* [19].

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